Mathematics

Complete Study Material with Explanations

BCA · First Year · Semester I

Units Covered	6 Units — 72 Hours
Topics	Set Theory · Logic · Matrices · Calculus · Numerics
Content	Theory · Definitions · Formulas · Solved Examples

UNIT I

Set Theory, Relations & Functions

12 Hours

1.1 Sets and Set Operations

A **set** is a well-defined collection of distinct objects called **elements** or **members**. Sets are denoted by capital letters A, B, C … and elements by lowercase letters a, b, c … If x is an element of A we write $x \in A$; if not, $x \notin A$.

Roster Form	Elements listed explicitly inside curly braces. E.g., A = {1, 2, 3, 4, 5}
Set-Builder Form	Elements described by a property. E.g., A = {x | x is a positive integer ≤ 5}

Types of Sets

Type	Definition	Example
Empty Set (∅)	No elements	∅ = {}
Singleton	Exactly one element	{5}
Finite Set	Countable elements	{a, e, i, o, u}
Infinite Set	Uncountably many	N = {1,2,3,…}
Equal Sets	Identical elements	{1,2}={2,1}
Equivalent Sets	Same cardinality	{1,2,3} ~ {a,b,c}
Universal Set (U)	All objects in context	U = {1..100}
Power Set P(A)	All subsets of A	A={1,2} → P(A)=4 subsets

Set Operations

Given two sets A and B defined over universal set U:

Operation	Symbol	Definition	Example (A={1,2,3}, B={2,3,4})
Union	A ∪ B	All elements in A or B	{1,2,3,4}
Intersection	A ∩ B	Elements in both A and B	{2,3}
Difference	A - B	Elements in A but not B	{1}
Complement	A' or Ac	Elements in U but not A	U - A
Sym. Difference	A ■ B	(A-B) ∪ (B-A)	{1,4}

De Morgan's Laws

$$(A \cup B)' = A' \cap B' \quad (A \cap B)' = A' \cup B'$$

These laws state that the complement of a union equals the intersection of complements and vice versa. Extremely useful for simplifying set expressions.

Example: Verify De Morgan's Law

Let U={1,2,3,4,5}, A={1,2,3}, B={2,3,4}

LHS: $(A\cup B)' = \{1,2,3,4\}' = \{5\}$

RHS: $A'\cap B' = \{4,5\}\cap\{1,5\} = \{5\}$ ✓ LHS = RHS

Cartesian Product

The **Cartesian product** A × B is the set of all ordered pairs (a,b) where a ∈ A and b ∈ B.

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$$

Example: Cartesian Product

A = {1,2}, B = {x,y}

A × B = {(1,x), (1,y), (2,x), (2,y)}

|A × B| = |A| × |B| = 2 × 2 = 4

1.2 Relations

A **relation** R from set A to set B is a subset of A × B. If (a,b) ∈ R we write aRb. The **domain** of R is {a | (a,b) ∈ R} and **range** is {b | (a,b) ∈ R}.

Properties of Relations (on a single set A)

Property	Definition	Example
Reflexive	aRa for all a ∈ A	'≤' on integers
Irreflexive	NOT aRa for any a	'<' on integers
Symmetric	aRb ■ bRa	'=' on reals
Antisymmetric	aRb and bRa ■ a=b	'≤' on integers
Transitive	aRb and bRc ■ aRc	'<' on naturals

Equivalence Relation	A relation that is Reflexive, Symmetric, AND Transitive. E.g., congruence modulo n: a ≡ b (mod n)

Example: Equivalence Relation

R = {(1,1),(2,2),(3,3),(1,2),(2,1)} on A={1,2,3}

Reflexive: (1,1),(2,2),(3,3) ∈ R ✓

Symmetric: (1,2)∈R ■ (2,1)∈R ✓

Transitive: (1,2)∈R, (2,1)∈R ■ (1,1)∈R ✓ → Equivalence Relation

1.3 Functions

A **function** f: A → B is a special relation where every element of A is related to **exactly one** element of B. A is called the **domain** and B the **co-domain**. The actual set of outputs is the **range**.

Type	Definition	Condition
One-One (Injective)	Different inputs → different outputs	f(a)=f(b) ■ a=b
Onto (Surjective)	Every element of B is an output	Range = Co-domain
Bijective	Both one-one and onto	Perfect pairing
Constant	All inputs → same output	f(x) = c
Identity	Each element maps to itself	f(x) = x

Example: Inverse Function

f: R → R, f(x) = 2x + 3

To find f■^{1}: let y = 2x+3 → x = (y-3)/2

Therefore f■^{1}(y) = (y-3)/2, and f■^{1}(f(x)) = x ✓

UNIT II Mathematical Logic & Propositional Calculus

10 Hours

2.1 Propositions and Logical Connectives

A **proposition** (or statement) is a declarative sentence that is either **TRUE (T)** or **FALSE (F)** — not both. E.g., '2+2=4' is TRUE; 'Delhi is in China' is FALSE. Questions, exclamations, and paradoxes are NOT propositions.

Connective	Symbol	Read as	True when
Negation	$\neg p$	NOT p	p is False
Conjunction	$p \wedge q$	p AND q	Both p, q are True
Disjunction	$p \vee q$	p OR q	At least one True
Implication	$p \rightarrow q$	If p then q	p=T,q=F is the only False case
Biconditional	$p \leftrightarrow q$	p if and only if q	p and q have same truth value

Truth Table for $p \rightarrow q$

The implication is the trickiest connective. 'If it rains, the ground gets wet.' This is **only FALSE** when it rains (p=T) but ground is NOT wet (q=F).

p	q	$\neg p$	$p \rightarrow q$	$p \leftrightarrow q$	$p \wedge q$	$p \vee q$
T	T	F	T	T	T	T
T	F	F	F	F	F	T
F	T	T	T	F	F	T
F	F	T	T	T	F	F

Tautology, Contradiction and Contingency

A **tautology** is a formula that is ALWAYS True (e.g., $p \vee \neg p$ — Law of Excluded Middle). A **contradiction** is always False (e.g., $p \wedge \neg p$). A **contingency** is sometimes True, sometimes False.

Logical Equivalence Laws

Law	Expression
Identity	$p \wedge T \equiv p$; $p \vee F \equiv p$
Domination	$p \vee T \equiv T$; $p \wedge F \equiv F$
Idempotent	$p \vee p \equiv p$; $p \wedge p \equiv p$
Double Negation	$\neg(\neg p) \equiv p$
Commutative	$p \vee q \equiv q \vee p$; $p \wedge q \equiv q \wedge p$

Associative	$(p \lor q) \lor r \equiv p \lor (q \lor r)$
Distributive	$p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$
De Morgan's	$\neg(p \land q) \equiv \neg p \lor \neg q$; $\neg(p \lor q) \equiv \neg p \land \neg q$
Absorption	$p \lor (p \land q) \equiv p$; $p \land (p \lor q) \equiv p$

2.2 Predicate Logic and Quantifiers

Propositional logic cannot express statements like 'All men are mortal.' Predicate logic uses **predicates** and **quantifiers** for this.

Universal Quantifier (∀)	'For all x, P(x) is true.' Written $\forall x\, P(x)$. Example: $\forall x\, (x+0=x)$
Existential Quantifier (∃)	'There exists at least one x such that P(x) is true.' Written $\exists x\, P(x)$. Example: $\exists x\, (x^2=4)$

Negation of Quantifiers

$$\neg(\forall x\, P(x)) \equiv \exists x\, \neg P(x) \quad \neg(\exists x\, P(x)) \equiv \forall x\, \neg P(x)$$

Example: Negation

Original: 'All cats are black' → $\forall x\, (Cat(x) \rightarrow Black(x))$

Negation: 'There exists a cat that is not black' → $\exists x\, (Cat(x) \land \neg Black(x))$

2.3 Rules of Inference

Rule	Form	Example
Modus Ponens	$p,\ p \rightarrow q\ \therefore\ q$	p: 'It rains'; p→q: 'If rain, wet'; ∴ wet
Modus Tollens	$\neg q,\ p \rightarrow q\ \therefore\ \neg p$	¬q: 'Not wet'; ∴ 'Did not rain'
Hyp. Syllogism	$p \rightarrow q,\ q \rightarrow r\ \therefore\ p \rightarrow r$	Chain of implications
Disjunctive Syll.	$p \lor q,\ \neg p\ \therefore\ q$	Either A or B; not A; so B
Addition	$p\ \therefore\ p \lor q$	True prop. → true disjunction
Simplification	$p \land q\ \therefore\ p$	From conjunction, extract one

2.4 Normal Forms (DNF and CNF)

Every propositional formula can be reduced to a standard form using logical equivalences.

DNF (Disjunctive Normal Form)	A disjunction (OR) of one or more conjunctions (AND) of literals. E.g., $(p \land q) \lor (\neg p \land r) \lor (q \land \neg r)$ is in DNF
CNF (Conjunctive Normal Form)	A conjunction (AND) of one or more clauses, each a disjunction of literals. E.g., $(p \lor q) \land (\neg p \lor r) \land (q \lor \neg r)$ is in CNF

UNIT III Matrices & Determinants 14 Hours

3.1 Introduction to Matrices

A **matrix** is a rectangular array of numbers arranged in rows and columns. A matrix with m rows and n columns is called an **m × n matrix**. The element at row i, column j is denoted a_{ij}.

$$A = [a_{ij}]_{m\times n} \text{ where } 1 \le i \le m,\ 1 \le j \le n$$

Types of Matrices

Type	Description	Condition
Row Matrix	Single row	m = 1
Column Matrix	Single column	n = 1
Square Matrix	Equal rows & cols	m = n
Diagonal Matrix	Non-diagonal = 0	a_{ij}=0 for i≠j
Scalar Matrix	Diagonal with equal values	a_{ij}=k·δ_{ij}
Identity Matrix (I)	Diagonal = 1, rest = 0	a_{ij}=1 if i=j
Zero/Null Matrix	All elements = 0	a_{ij}=0 ∀i,j
Symmetric	A = A<super>T</super>	a_{ij} = a_{ji}
Skew-Symmetric	A = -A<super>T</super>	a_{ij} = -a_{ji}
Orthogonal	A · A<super>T</super> = I	Inverse = Transpose

Matrix Operations

Addition: $(A+B)_{ij} = a_{ij} + b_{ij}$ (only for same-order matrices)

Scalar Multiplication: $(kA)_{ij} = k \cdot a_{ij}$

Matrix Multiplication: For A(m×n) and B(n×p), C = A·B is m×p, where $C_{ij} = \Sigma\, a_{ik} \cdot b_{kj}$

Note: *Matrix multiplication is NOT commutative: AB ≠ BA in general.*

3.2 Determinants

The **determinant** is a scalar value computed from a square matrix that encodes geometric and algebraic information about the matrix.

Determinant of 2×2 Matrix

$$\text{If } A = [[a,b],[c,d]] \text{ then } \det(A) = |A| = ad - bc$$

Example: 2×2 Determinant

A = [[3,5],[1,2]]

|A| = (3)(2) - (5)(1) = 6 - 5 = 1

Determinant of 3×3 Matrix (Sarrus' Rule)

For a 3×3 matrix A = [[a_{11}…]]:

$$|A| = a_{11}(a_{22}a_{33}-a_{23}a_{32}) - a_{12}(a_{21}a_{33}-a_{23}a_{31}) + a_{13}(a_{21}a_{32}-a_{22}a_{31})$$

Example: 3×3 Determinant

A = [[1,2,3],[4,5,6],[7,8,9]]

|A| = 1(5·9-6·8) - 2(4·9-6·7) + 3(4·8-5·7)

= 1(45-48) - 2(36-42) + 3(32-35)

= 1(-3) - 2(-6) + 3(-3) = -3 + 12 - 9 = 0

Properties of Determinants

- If any row/column is all zeros → det = 0
- Swapping two rows/columns changes sign of det
- If two rows/columns are identical → det = 0
- Multiplying a row by k multiplies det by k
- det(AB) = det(A)·det(B)
- $\det(A^T) = \det(A)$
- $\det(A^{-1}) = 1/\det(A)$
- $\det(kA) = k^n \cdot \det(A)$ for n×n matrix

3.3 Inverse of a Matrix

The **inverse** of a square matrix A (denoted A^{-1}) satisfies $A \cdot A^{-1} = A^{-1} \cdot A = I$. It exists only if det(A) ≠ 0 (i.e., A is **non-singular**).

$$A^{-1} = (1/\det(A)) \times \text{adj}(A) \text{ where adj}(A) = \text{Transpose of Cofactor Matrix}$$

Example: Finding Inverse of 2×2 Matrix

A = [[2,1],[5,3]]

det(A) = 2·3 - 1·5 = 6 - 5 = 1

adj(A) = [[3,-1],[-5,2]]

A■¹ = (1/1)·[[3,-1],[-5,2]] = [[3,-1],[-5,2]]

Verify: A·A■¹ = [[2,1],[5,3]]·[[3,-1],[-5,2]] = [[1,0],[0,1]] = I ✓

3.4 System of Linear Equations

A system AX = B can be solved by: Cramer's Rule, Gaussian Elimination, or Row Reduction to Echelon Form.

Cramer's Rule (for n×n system)

$$x_i = \det(A_i) / \det(A) \text{ where } A_i \text{ is A with i-th column replaced by B}$$

Example: Cramer's Rule — 2×2 System

2x + y = 5 ; x + 3y = 10

A = [[2,1],[1,3]], B = [[5],[10]]

det(A) = 6-1 = 5

A_1 = [[5,1],[10,3]] → det = 15-10 = 5 → x = 5/5 = 1

A_2 = [[2,5],[1,10]] → det = 20-5 = 15 → y = 15/5 = 3

Solution: x=1, y=3

Rank of a Matrix

Rank	The rank of matrix A is the maximum number of linearly independent rows (or columns). Rank(A) = r means every (r+1)×(r+1) sub-matrix has det = 0, but some r×r sub-matrix has det ≠ 0.

3.5 Eigenvalues and Eigenvectors

For a square matrix A, a scalar λ and non-zero vector X satisfying **AX = λX** are called an **eigenvalue** and its corresponding **eigenvector**.

The eigenvalues are roots of the characteristic equation:

det(A - λI) = 0 ← Characteristic Equation

Example: Eigenvalue Calculation

A = [[4,1],[2,3]]

det(A-λI) = det([[4-λ,1],[2,3-λ]]) = (4-λ)(3-λ) - 2 = 0

= λ^2-7λ+12-2 = λ^2-7λ+10 = 0

λ = (7 ± √(49-40))/2 = (7±3)/2 → λ■ = 5, λ■ = 2

For λ■=5: (A-5I)X=0 → [[-1,1],[2,-2]]X=0 → X = $[1,1]^T$

For λ■=2: (A-2I)X=0 → [[2,1],[2,1]]X=0 → X = $[1,-2]^T$

Cayley-Hamilton Theorem	Every square matrix satisfies its own characteristic equation. If p(λ) = 0 is the characteristic equation, then p(A) = 0 (zero matrix). Used to compute A■¹ without row reduction.

UNIT IV Differential Calculus

14 Hours

4.1 Limits

The **limit** of f(x) as x approaches a is the value f(x) approaches (but need not reach). Written: $\lim_{x\to a} f(x) = L$

$\lim_{x\to a} f(x) = L$ ■ for every ε>0, ∃ δ>0 such that |x-a|<δ ■ |f(x)-L|<ε

Standard Limits

Formula	Value
lim_{x→0} (sin x)/x	1
lim_{x→0} (1-cos x)/x	0
lim_{x→0} (e<super>x</super>-1)/x	1
lim_{x→0} (a<super>x</super>-1)/x	ln a
lim_{x→∞} (1+1/x)<super>x</super>	e
lim_{x→a} (x<super>n</super>-a<super>n</super>)/(x-a)	na<super>n-1</super>
lim_{x→0} (tan x)/x	1
lim_{x→0} (log(1+x))/x	1

4.2 Continuity

A function f is **continuous at x=a** if all three conditions hold:

1. f(a) is defined
2. $\lim_{x\to a} f(x)$ exists (Left-hand limit = Right-hand limit)
3. $\lim_{x\to a} f(x) = f(a)$

Example: Checking Continuity

$f(x) = x^2$ at x=3

f(3) = 9 ✓ (defined)

$\lim_{x\to 3} x^2 = 9$ ✓ (limit exists)

lim = f(3) = 9 ✓ → f is continuous at x=3

4.3 Differentiation

The **derivative** of f at x is the limit of the difference quotient, representing the instantaneous rate of change or slope of the tangent line.

$$f'(x) = \lim_{h\to 0} [f(x+h) - f(x)] / h$$

Standard Derivatives

f(x)	f'(x)	tan x	sec<super>2</super>x
x<super>n</super>	nx<super>n-1</super>	cot x	-cosec<super>2</super>x
e<super>x</super>	e<super>x</super>	sec x	sec x tan x
a<super>x</super>	a<super>x</super> ln a	cosec x	-cosec x cot x
ln x	1/x	sin<super>-1</super>x	1/√(1-x<super>2</super>)
log_ax	1/(x ln a)	tan<super>-1</super>x	1/(1+x<super>2</super>)
sin x	cos x	√x	1/(2√x)
cos x	-sin x		

Rules of Differentiation

Rule	Formula
Sum/Difference	d/dx[f±g] = f'±g'
Constant Multiple	d/dx[cf] = c·f'
Product Rule	d/dx[fg] = f'g + fg'
Quotient Rule	d/dx[f/g] = (f'g - fg') / g<super>2</super>
Chain Rule	d/dx[f(g(x))] = f'(g(x))·g'(x)
Implicit Diff.	Differentiate both sides; isolate dy/dx
Logarithmic Diff.	Take ln both sides, then differentiate
Parametric Diff.	dy/dx = (dy/dt)/(dx/dt)

Example: Product Rule and Chain Rule

Find $d/dx[(x^2+1)\cdot\sin x]$

Product Rule: $(2x)(\sin x) + (x^2+1)(\cos x)$

$= 2x\cdot\sin x + (x^2+1)\cdot\cos x$

Find $d/dx[\sin(x^2)]$

Chain Rule: $\cos(x^2) \cdot 2x = 2x\cdot\cos(x^2)$

4.4 Applications of Derivatives

Rolle's Theorem: If f is continuous on [a,b], differentiable on (a,b), and f(a)=f(b), then there exists c ∈ (a,b) such that f'(c)=0.

Mean Value Theorem (Lagrange): If f is continuous on [a,b] and differentiable on (a,b), then there exists c ∈ (a,b) such that:

$$f'(c) = [f(b) - f(a)] / (b - a)$$

Maxima and Minima

First Derivative Test: f has a local max at c if $f'(c)=0$ and f' changes from + to -. Local min if f' changes from - to +.

Second Derivative Test: If $f'(c)=0$ and $f''(c)>0 \to$ local min; $f''(c)<0 \to$ local max.

Example: Find extrema of $f(x) = x^3 - 3x + 2$

$f'(x) = 3x^2-3 = 0 \to x^2=1 \to x=\pm 1$

$f''(x) = 6x$

$x=1$: $f''(1)=6>0 \to$ Local MINIMUM, $f(1)=1-3+2=0$

$x=-1$: $f''(-1)=-6<0 \to$ Local MAXIMUM, $f(-1)=-1+3+2=4$

L'Hôpital's Rule

If $\lim f(x)/g(x)$ gives 0/0 or ∞/∞ (indeterminate form), then:

$$\lim_{x\to a} f(x)/g(x) = \lim_{x\to a} f'(x)/g'(x) \text{ (if the latter limit exists)}$$

Example: L'Hôpital's Rule

$\lim_{x\to 0} \sin x / x = 0/0$ form

Apply: $\lim_{x\to 0} \cos x / 1 = \cos(0) / 1 = 1$ ✓

UNIT V Integral Calculus 12 Hours

5.1 Indefinite Integration

Integration is the reverse process of differentiation. The **indefinite integral** of f(x) is a family of functions F(x)+C where F'(x)=f(x). C is the constant of integration.

$$\int f(x)dx = F(x) + C \text{ where } F'(x) = f(x)$$

Standard Integrals

∫f(x)dx	Result	∫sec<super>2</super>x dx	tan x + C
∫x<super>n</super>dx	x<super>n+1</super>/(n+1) + C, n≠-1	∫cosec<super>2</super>x dx	-cot x + C
∫1/x dx	ln|x| + C	∫sec x tan x dx	sec x + C
∫e<super>x</super>dx	e<super>x</super> + C	∫1/√(1-x<super>2</super>) dx	sin<super>-1</super>x + C
∫a<super>x</super>dx	a<super>x</super>/ln a + C	∫1/(1+x<super>2</super>) dx	tan<super>-1</super>x + C
∫sin x dx	-cos x + C	∫1/(x<super>2</super>-a<super>2</super>) dx	(1/2a)ln|(x-a)/(x+a)| + C
∫cos x dx	sin x + C		

Methods of Integration

1. Integration by Substitution: Replace a complicated expression with a single variable u.

Example: Substitution

$\int 2x\cdot\cos(x^2)dx$

Let $u=x^2$, then du=2x dx

$= \int\cos(u)du = \sin(u)+C = \sin(x^2)+C$

2. Integration by Parts (ILATE Rule): Used when integrand is a product of two functions.

∫u·v dx = u·∫v dx - ∫(u'·∫v dx) dx (ILATE: choose u as Inverse, Log, Algebraic, Trig, Exponential)

Example: Integration by Parts

$\int x\cdot e^x dx$

u=x (algebraic), $v=e^x$ (exponential)

$= x\cdot e^x - \int 1\cdot e^x dx$

$= x\cdot e^x - e^x + C = e^x(x-1) + C$

3. Partial Fractions: Decompose a rational function into simpler fractions before integrating.

Example: Partial Fractions

$\int 1/(x^2-1)\,dx = \int 1/((x-1)(x+1))\,dx$

$1/((x-1)(x+1)) = A/(x-1) + B/(x+1)$

$\rightarrow A=1/2, B=-1/2$

$= (1/2)\int 1/(x-1)dx - (1/2)\int 1/(x+1)dx$

$= (1/2)\ln|x-1| - (1/2)\ln|x+1| + C = (1/2)\ln|(x-1)/(x+1)| + C$

5.2 Definite Integration

A **definite integral** gives a numerical value representing the net area under a curve between two limits.

$$\int_a^b f(x)dx = F(b) - F(a) \text{ (Fundamental Theorem of Calculus)}$$

Properties of Definite Integrals

- $\int_a^b f(x)dx = -\int_b^a f(x)dx$
- $\int_a^a f(x)dx = 0$
- $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$
- $\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx$ if $f(2a-x)=f(x)$
- $\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$ if f is even; 0 if f is odd

Example: Definite Integral

$\int_0^{\pi/2} \sin x\,dx = [-\cos x]_0^{\pi/2}$

$= -\cos(\pi/2) - (-\cos(0)) = 0 - (-1) = 1$

5.3 Applications of Integration

Area under a curve:

$$\text{Area} = \int_a^b |f(x)|\,dx$$

Area between two curves:

$$\text{Area} = \int_a^b [f(x) - g(x)]\,dx \text{ where } f(x) \geq g(x) \text{ on } [a,b]$$

Volume of Revolution (Disk Method):

$$V = \pi \int_a^b [f(x)]^2\,dx \text{ (rotation about x-axis)}$$

UNIT VI Numerical Methods 10 Hours

6.1 Numerical Solution of Equations

Many algebraic and transcendental equations (e.g., $e^x=3x$ or $x^3-x-1=0$) cannot be solved analytically. Numerical methods give approximate solutions to any desired accuracy.

Bisection Method

If $f(a)\cdot f(b) < 0$, there is a root in [a,b]. The method repeatedly bisects the interval.

Algorithm:

1. Find a, b such that $f(a)\cdot f(b) < 0$
2. Compute midpoint $c = (a+b)/2$
3. If $f(c)=0$ or $|b-a|/2 <$ tolerance: c is the root
4. If $f(a)\cdot f(c) < 0$: root is in [a,c] → set b=c; else set a=c
5. Repeat from step 2

Convergence: Error $\le (b-a)/2^n$ after n iterations

Example: Bisection Method

Find root of $f(x)=x^3-x-1=0$ in [1,2]

$f(1)=1-1-1=-1(<0)$, $f(2)=8-2-1=5(>0)$ → root in [1,2]

$c=(1+2)/2=1.5$ → $f(1.5)=3.375-1.5-1=0.875(>0)$ → root in [1,1.5]

$c=(1+1.5)/2=1.25$ → $f(1.25)=1.953-1.25-1=-0.297(<0)$ → root in [1.25,1.5]

$c=1.375$ → $f(1.375)\approx 0.224>0$ → root in [1.25,1.375]

Continue until desired precision. Root ≈ 1.3247

Newton-Raphson Method

Faster (quadratic) convergence by using tangent lines. Requires f'(x) to be computed.

$$x_{n+1} = x_n - f(x_n) / f'(x_n)$$

Example: Newton-Raphson

Solve $x^3-x-1=0$, start $x_0=1.5$

$f(x)=x^3-x-1$, $f'(x)=3x^2-1$

$x_1=1.5 - (1.5^3-1.5-1)/(3\cdot 1.5^2-1)$

$= 1.5 - (3.375-2.5)/(6.75-1) = 1.5 - 0.875/5.75 \approx 1.3478$

$x_2=1.3478-f(1.3478)/f'(1.3478) \approx 1.3252$ (converging rapidly)

Regula-Falsi Method

Also called False Position Method. Uses the x-intercept of the secant line.

$$x_2 = [a \cdot f(b) - b \cdot f(a)] / [f(b) - f(a)]$$

6.2 Interpolation

Interpolation is estimating the value of a function at intermediate points using known data values.

Finite Differences

Forward Difference (Δ)	$\Delta f(x) = f(x+h) - f(x)$. Higher orders: $\Delta^2 f(x) = \Delta(\Delta f(x)) = f(x+2h)-2f(x+h)+f(x)$
Backward Difference (∇)	$\nabla f(x) = f(x) - f(x-h)$. Used at the end of a table.

Newton's Forward Interpolation Formula

f(x■+sh) = f(x■) + s·Δf(x■) + [s(s-1)/2!]·Δ²f(x■) + [s(s-1)(s-2)/3!]·Δ³f(x■) + … where s=(x-x■)/h

Used when interpolating near the **beginning** of the table.

Newton's Backward Interpolation Formula

$$f(x_n+sh) = f(x_n) + s \cdot \nabla f(x_n) + [s(s+1)/2!] \cdot \nabla^2 f + \ldots \text{ where } s=(x-x_n)/h$$

Used when interpolating near the **end** of the table.

Lagrange's Interpolation Formula

Works for **unequally spaced** data. For n+1 data points:

$$P(x) = \Sigma f(x_i) \cdot L_i(x) \text{ where } L_i(x) = \Pi (x-x_j)/(x_i-x_j),\ j \neq i$$

Example: Lagrange Interpolation

Data: (1,1), (2,8), (4,64) → Estimate f(3)

$L_0(3) = (3-2)(3-4)/((1-2)(1-4)) = (1)(-1)/((-1)(-3)) = -1/3$

$L_1(3) = (3-1)(3-4)/((2-1)(2-4)) = (2)(-1)/((1)(-2)) = 1$

$L_2(3) = (3-1)(3-2)/((4-1)(4-2)) = (2)(1)/((3)(2)) = 1/3$

$f(3) \approx 1 \cdot (-1/3) + 8 \cdot (1) + 64 \cdot (1/3) = -1/3 + 8 + 64/3 = 27$ ✓ (matches $3^3=27$)

6.3 Numerical Integration

When an integral cannot be evaluated analytically, numerical methods approximate the area under a curve.

Trapezoidal Rule

$$\int_a^b f(x)dx \approx (h/2)[y_0 + 2(y_1+y_2+\ldots+y_{n-1}) + y_n] \text{ where } h=(b-a)/n$$

Simpson's 1/3 Rule (n must be even)

$$\int_a^b f(x)dx \approx (h/3)[y_0 + 4(y_1+y_3+\ldots) + 2(y_2+y_4+\ldots) + y_n]$$

Simpson's 3/8 Rule (n must be multiple of 3)

$$\int_a^b f(x)dx \approx (3h/8)[y_0 + 3(y_1+y_2+y_4+\ldots) + 2(y_3+y_6+\ldots) + y_n]$$

Example: Trapezoidal Rule

Evaluate $\int_0^1 1/(1+x)\,dx$ using n=4

h=(1-0)/4=0.25; x: 0, 0.25, 0.5, 0.75, 1

y: 1, 0.8, 0.6667, 0.5714, 0.5

= (0.25/2)[1 + 2(0.8+0.6667+0.5714) + 0.5]

= 0.125[1 + 2(2.0381) + 0.5] = 0.125 × 5.5762 ≈ 0.6970

Exact: ln(2) ≈ 0.6931 (close!)

6.4 Runge-Kutta Method (4th Order)

Solves ordinary differential equations dy/dx=f(x,y) numerically without computing higher derivatives. It is the most widely used method due to high accuracy.

$$k_1=h\cdot f(x_n,y_n)\ k_2=h\cdot f(x_n+h/2, y_n+k_1/2)\ k_3=h\cdot f(x_n+h/2, y_n+k_2/2)\ k_4=h\cdot f(x_n+h, y_n+k_3)\ y_{n+1} = y_n + (1/6)(k_1+2k_2+2k_3+k_4)$$

Example: RK4 Method

Solve dy/dx = x+y, y(0)=1, find y(0.1) with h=0.1

k_1 = 0.1·f(0,1) = 0.1(0+1) = 0.1

k_2 = 0.1·f(0.05, 1.05) = 0.1(0.05+1.05) = 0.11

k_3 = 0.1·f(0.05, 1.055) = 0.1(0.05+1.055) = 0.1105

k_4 = 0.1·f(0.1, 1.1105) = 0.1(0.1+1.1105) = 0.12105

y(0.1) = 1 + (1/6)(0.1+2(0.11)+2(0.1105)+0.12105)

= 1 + (1/6)(0.6621) ≈ 1.1103

Comparison of Numerical Methods

Method	Order	Convergence	Best For
Bisection	1st	Linear (slow)	Guaranteed root in [a,b]
Regula-Falsi	1st	Linear (faster)	Smoother convergence
Newton-Raphson	2nd	Quadratic (fast!)	Good initial guess available
Trapezoidal Rule	$O(h^2)$	2nd order	Simple, quick estimation
Simpson's 1/3	O(h■)	4th order	Higher accuracy, even n
RK4	O(h■)	4th order	ODE solving, general purpose

Quick Formula Reference Sheet

SET THEORY	
	• $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$
	• $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$
	• De Morgan: $(A \cup B)' = A' \cap B'$
	• $\lvert P(A)\rvert = 2^{\lvert A\rvert}$
	• $\lvert A \cup B\rvert = \lvert A\rvert + \lvert B\rvert - \lvert A \cap B\rvert$

LOGIC	
	• $p \to q \equiv \neg p \lor q$
	• $p \leftrightarrow q \equiv (p \to q) \land (q \to p)$
	• $\neg(p \land q) \equiv \neg p \lor \neg q$ (De Morgan)
	• $\neg \forall x P(x) \equiv \exists x \neg P(x)$

MATRICES	
	• $\lvert AB\rvert = \lvert A\rvert \cdot \lvert B\rvert$
	• $A^{-1} = \text{adj}(A)/\lvert A\rvert$
	• $AX = \lambda X$ (eigenvalue equation)
	• $\det(A - \lambda I) = 0$ (char. equation)

DIFFERENTIATION	
	• $d/dx(x^n) = nx^{n-1}$
	• $d/dx(e^x) = e^x$
	• $d/dx(\ln x) = 1/x$
	• Product: $(uv)' = u'v + uv'$
	• Chain: $d/dx[f(g)] = f'(g) \cdot g'$

INTEGRATION	
	• $\int x^n dx = x^{n+1}/(n+1) + C$
	• $\int e^x dx = e^x + C$
	• $\int \sin x \, dx = -\cos x + C$
	• $\int$by parts: $\int uv' = uv - \int u'v$
	• FTC: $\int_a^b f \, dx = F(b) - F(a)$

NUMERICAL METHODS	

	• Bisection: c=(a+b)/2
	• Newton-Raphson: $x_{n+1}=x_n-f/f'$
	• Trapezoid: $(h/2)[y_0+2\Sigma y_i+y_n]$
	• Simpson 1/3: $(h/3)[y_0+4odd+2even+y_n]$